1. The exam will have questions emphasizing the material in sections 2.6–2.7, 3.1–3.3, 4.1–4.4 of the text.

2. There is no time limit for the exam. Most people will probably need at least 2 hours. It would be a good idea allow for extra time just in case you need it.

3. Books, notes, and calculators are not allowed.

4. To receive full credit you should show your work and write neatly. The solutions should be clearly presented. Try to use correct spelling and punctuation.

Specific Preparation and Sample Problems for Midterm 2

1. Memorize all of the definitions and theorems. (Theorem numbers are not important, but it is important to remember which theorems come before other theorems.) Be able to give simple examples that illustrate definitions or theorems.

2. ★ There will be a question from earlier in the semester about directly using the definition of the limit to prove certain limits. In these proofs, you should not use any theorems about limits, but directly rely on the definition. Examples:

   (a) Suppose \( \lim(x_n) = 3 \). Directly use the definition of the limit of a sequence to prove that \( \lim\left(\frac{x_n^2 + 3x_n + 7}{2x_n}\right) = \frac{25}{6} \). You should not use the Algebraic Limit Theorem or other theorems about limits in your argument.

   (b) Suppose \( \lim(a_n) = -2 \). Directly use the definition of the limit of a sequence to prove that \( \lim\left(\frac{a_n+4}{a_n-4}\right) = -\frac{1}{3} \). You should not use the Algebraic Limit Theorem or other theorems about limits in your argument.

3. ★ There will be a question testing the ability to use the definition of a Cauchy sequence. Examples of problems of this type are

   (a) Suppose \((x_n)\) is a Cauchy sequence such that \(3 < x_n < 10\) for all \(n \in \mathbb{N}\). By directly using the definition of a Cauchy sequence, show that \(\left(\frac{x_n+2}{x_n-1}\right)\) is a Cauchy sequence. Your argument may not use Cauchy’s Criterion or other theorems about limits.

   (b) Suppose \((a_n)\) is a Cauchy sequence and suppose \(A \leq a_n \leq B\) for all \(n \in \mathbb{N}\), where \(A\) and \(B\) are real numbers (which are not necessarily positive). By directly using the definition of a Cauchy sequence, show that \((a_n^2 - a_n + 7)\) is a Cauchy sequence. Your argument may not use Cauchy’s Criterion or other theorems about limits.

4. You should be able to answer any of the following exercises:

   2.6.2, 2.7.11, 3.2.6, 3.2.10, 3.3.2, 3.3.4, 4.2.2, 4.2.5, 4.3.8

5. Understand the meaning of Theorem 2.7.10 (Rearrangements of an absolutely convergent series)
6. There is likely to be a question similar to one of Section 2.7 Exercises 1–4.

7. Understand the meaning of Theorem 2.8.1.

8. * There will be a question involving a rational function whose limit you will need to compute directly from the definition without using the Algebraic Limit Theorem. Example: Directly use the \( \epsilon - \delta \) definition of the limit to show that \( \lim_{x \to 3} \frac{1}{x^2} = \frac{1}{9} \). Do not use the Algebraic Limit Theorem in your proof.

9. * Another example like the previous item: Directly use the \( \epsilon - \delta \) definition of the limit to show that \( \lim_{x \to 2} \frac{x^3}{x^2 + 1} = \frac{8}{5} \). Do not use the Algebraic Limit Theorem in your proof.

10. Understand the meaning of Theorem 4.2.3 (Sequential Criterion for Functional Limits)
    Understand Corollary 4.2.4 (Algebraic Limit Theorem for Functional Limits), and know how to prove it using Theorem 4.2.3 assuming you already know the Algebraic Limit Theorem for sequences.

11. If \( A \) and \( B \) are open subsets of \( \mathbb{R} \), show that \( A \cup B \) and \( A \cap B \) are open.

12. If \( A \) and \( B \) are closed subsets of \( \mathbb{R} \), show that \( A \cup B \) and \( A \cap B \) are closed.

13. Be able to prove De Morgan’s Law’s (Exercise 3.2.9): Given a collection of sets \( \{E_\lambda : \lambda \in \Lambda\} \), show that
    \[
    \left( \bigcup_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c \quad \text{and} \quad \left( \bigcap_{\lambda \in \Lambda} E_\lambda \right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c.
    \]

14. Prove that if \( \{\mathcal{O}_\lambda : \lambda \in \Lambda\} \) is any collection of open sets, then \( \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda \) is open. [Note that the indexing set \( \Lambda \) potentially could be uncountable and so it is not appropriate to use integer subscripts in this proof.] What about the intersection of finitely many open sets?

15. Prove that if \( \{F_\lambda : \lambda \in \Lambda\} \) is any collection of closed sets, then \( \bigcap_{\lambda \in \Lambda} F_\lambda \) is closed. What about the union of finitely many closed sets?

16. Give an explicit example of open subsets \( A_n \) for \( n \in \mathbb{N} \) such that \( \bigcap_{n=1}^\infty A_n \) is not open.

17. Give an explicit example of closed subsets \( A_n \) for \( n \in \mathbb{N} \) such that \( \bigcup_{n=1}^\infty A_n \) is not closed.

18. Prove that a set \( \mathcal{O} \) is open if and only if \( \mathcal{O}^c \) is closed. Likewise, a set \( F \) is closed if and only if \( F^c \) open.

19. * You will need to prove a fact about continuous functions. Good theorems to study are:

   (a) Theorem 4.3.2
   (b) Corollary 4.3.3
   (c) Examples 4.3.5–4.3.8
   (d) Theorem 4.3.9 (Do this directly from the definition of continuity.)
(e) If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( B \subseteq \mathbb{R} \) is open, prove that \( f^{-1}(B) = \{ x \in \mathbb{R} : f(x) \in B \} \) is open.

20. Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at all points in \( \mathbb{R} \) and suppose \( f(3) = 5 \).
   
   (a) Prove there exists an open neighborhood \( V_{\delta}(3) \) such that \( 1 < f(x) < 8 \) for all \( x \in V_{\delta}(3) \).
   
   (b) Prove that there exist an open neighborhood \( V_{\delta'}(3) \) such that \( \frac{1}{11} < \frac{1}{f(x)} < 17 \) for all \( x \in V_{\delta'}(3) \).

21. By directly using the \( \epsilon-\delta \) definition of continuity, prove that \( f(x) = \sqrt{x} \) is continuous at \( x = 9 \). See Example 4.3.8. See Exercise 4.3.1.

22. Know the definition of a compact set, understand the Characterization of Compactness in \( \mathbb{R} \) (Thm. 3.3.4), the Nested Compact Set Property (Thm. 3.3.5), open covers, and the Heine-Borel Theorem (Thm. 3.3.8).

23. Exercise 3.3.2 on deciding which sets are compact and why.

24. * There will be at least one question dealing with uniform continuity.

25. If \( f \) is uniformly continuous on the interval \((a, b]\) and also uniformly continuous on \([b, c)\) where \( a < b < c \), prove that \( f \) is uniformly continuous on \((a, c]\).

26. Be able to answer any of exercises, 4.4.3, 4.4.4, 4.4.6, 4.4.8.

27. (a) Suppose that \( \lim_{x \to c} f(x) = A \neq 0 \). Using the \( \epsilon-\delta \) definition of the limit, prove that \( \lim_{x \to c} \frac{1}{f(x)} = \frac{1}{A} \).
   
   (b) Suppose that \( \lim_{x \to c} f(x) = A \) and \( \lim_{x \to c} g(x) = B \). Use the \( \epsilon-\delta \) definition of the limit to prove that \( \lim_{x \to c} f(x)g(x) = AB \). [Hint: Don’t divide by zero!]

28. Theorem 3.2.13: Prove that \( G \) is open if and only if \( G^c \) is closed. Prove that \( F \) is closed if and only if \( F^c \) is open.