Math 341 Section 2 – Winter 2022 – Midterm 1 Review Sheet

1. This exam is in the Testing Center on Wed, Feb 2 and Thu, Feb 3.
2. The exam will have questions based on material in sections 1.1-1.6 and 2.1-2.6 of the text.
3. No books or notes.
4. To receive full credit you should show your work and write neatly. The solutions should be clearly presented. Try to use correct spelling and punctuation. Do your best to format and arrange your solutions nicely. Sometimes you may need to work a problem on scratch paper before writing the final solution on the exam sheet.
5. Useful study strategies:
   (a) Work exercises beyond what was required on the homework.
   (b) Read each section of the text at least 3 or 4 times.
   (c) Memorize the definitions of any important terminology as well as statements of theorems.
   (d) Understand 3 or 4 examples for each important concept or method.
   (e) Explain difficult proofs and solutions to each other.

Specific Preparation and Sample Problems for Midterm 1

⋆⋆ = “guaranteed to be on the test.” ⋆ = “likely to be on the test but not guaranteed.”

1. You should be able to correctly state the definitions of all of the following:
   bounded above, upper bound, least upper bound, supremum, bound below, lower bound, greatest lower bound, infimum, maximum of a set, minimum of a set, finite set, countable set, uncountable set, sequence, limit of a sequence, convergent sequence, bounded sequence, monotone sequence, subsequence, Cauchy sequence, convergent series.
   Be prepared to give simple examples that illustrate each of these definitions.

2. Classify each of the following sets as finite, countable, or uncountable.
   (a) \( \mathbb{N} \)
   (b) \( \mathbb{N} \times \mathbb{N} = \{(a, b) : a, b \in \mathbb{N}\} \)
   (c) \( \{x \in \mathbb{R} : x \notin \mathbb{Q}\} \)
   (d) The set \( S \) where \( S \) is the set of all functions whose domain is the set \( \{0, 1, 2\} \) and whose codomain is the set \( \mathbb{N} \).
   (e) The set \( T \) where \( T \) is the set of all functions whose domain is the set \( \mathbb{Q} \) and whose codomain is the set \( \{-1, 0, 1\} \).
   (f) The set of all rational numbers that can be expressed as \( p/q \) where \( p, q \in \mathbb{Z}, q \neq 0, \) and \( p+q \leq 10 \).

3. ** Part of or all of one of the following four questions is guaranteed be be on the exam:
   (a) Prove that \( \mathbb{R} \) is uncountable using the Nested Interval Theorem as in the proof of Theorem 1.5.6 (ii).
   (b) Prove that the interval \( (0, 1) \) is uncountable by Cantor’s diagonalization method as in the proof of Theorem 1.6.1.
(c) Exercise 1.6.4: Let $S$ be the set consisting of all sequences of 0’s and 1’s. That is, 
$$S = \{(a_1, a_2, a_3, \ldots): a_n = 0 \text{ or } 1\}.$$  
Give a rigorous argument showing that $S$ is uncountable.

(d) Let $A$ be a set and let $\mathcal{P}(A)$ denote the power set of $A$. Show that there does not exist a surjective function $f: A \to \mathcal{P}(A)$. That is, prove Theorem 1.6.2.

4. Let $(a_n) \to A$ and $(b_n) \to B$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Use the definition of the limit to show that $A \leq B$.

5. ★ Completing this exercise proves the existence of $\sqrt{3}$. Show that the sequence $(a_n)$ where $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{3}{a_n})$ converges and find its limit. See exercise 2.4.5.

6. ★ A question involving ideas similar to those in the proofs of the following two theorems:

(a) Prove the Bolzano-Weierstrass Theorem (Theorem 2.5.5): Every bounded sequence contains a convergent subsequence.

(b) Prove the Nested Interval Theorem (Theorem 1.4.1): For each $n$ assume $I_n$ is a closed interval of the form $I_n = [a_n, b_n]$ and assume that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$. Then the nested sequence of closed intervals 
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$$ 
has a nonempty intersection; that is $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

7. ★★ One of the following three questions is guaranteed to be on the exam:

(a) Let $(a_n)$ and $(b_n)$ be convergent sequences such that $\lim a_n = A$ and $\lim b_n = B$. By using the definition of the limit, show that 
$$\lim_{n \to \infty} (a_n + b_n) = A + B.$$ 

(b) Let $(a_n)$ and $(b_n)$ be convergent sequences such that $\lim a_n = A$ and $\lim b_n = B$. By using the definition of the limit, show that 
$$\lim_{n \to \infty} (a_n b_n) = AB.$$ 

(c) Let $(a_n)$ be a convergent sequence and assume $\lim a_n = A \neq 0$. By directly using the definition of the limit, prove that 
$$\lim_{n \to \infty} \frac{1}{a_n} = \frac{1}{A}.$$ 

8. ★★ One of the following four questions is guaranteed to be on the exam:

(a) Exercise 2.3.3 (Squeeze Theorem)

(b) Exercise 2.4.1

(c) Exercise 2.4.5

(d) Theorem 2.5.2

9. ★★ A limit problem somewhat like this is guaranteed to be on the exam: Use the definition of the limit (without quoting theorems about limits) to directly show that if $\lim_{n \to \infty} a_n = 5$, then 
$$\lim_{n \to \infty} \frac{a_n + 2}{a_n - 2} = \frac{7}{3} \text{ or } \lim_{n \to \infty} (2a_n^2 - a_n + 3) = 48. \text{ (Be careful about making sure } N \text{ is large enough so that the denominator is not zero when } n \geq N.)$$

Also be able to use the definition of the limit for examples like Exercise 2.2.2.
10. There may be a question asking whether certain statements are true or false, or you may be asked to work with concrete examples. Questions of this sort are

(a) Exercises 2.3.4, 2.3.7
(b) Exercise 2.5.1
(c) Exercises 2.4.1, 2.4.2, 2.4.3
(d) Let \( A = \{n/(n + m) : n, m \in \mathbb{N}\} \). Do \( \sup A \) and \( \inf A \) exist? If so, what are they? Does \( A \) have a maximum or minimum? If so, what are they?

11. Let \( f : \mathbb{R} \to [0, \infty) \) be given by the formula \( f(x) = \frac{1}{x^2 + 1} \).

(a) Find (and simplify) \( f(A) \) where \( A = [-1, 2], \ A = (-1, 2], \ A = [0, \infty) \).
(b) Find (and simplify) \( f^{-1}(B) \) where \( B = [0, 1], \ B = (0, 1), \ B = [0, \infty) \).

12. Let \( (a_n) \) be a bounded sequence with the property that every convergent subsequence \( (a_{k_n}) \) converges to the same limit \( A \in \mathbb{R} \). Show that \( (a_n) \) must converge to \( A \).

13. ⋆⋆ A question involving Cauchy sequences is guaranteed to be on the exam. Good examples to study are exercises 2.6.2, 2.6.3, 2.6.4 and Theorem 2.6.2. A kind of example you should be able to do is: Suppose \( (x_n) \) is a Cauchy sequence, then prove that the new sequence \( (\frac{x_{2n}^2}{x_n^{2n} + 1}) \) is also a Cauchy sequence. Or, assume \( (a_n) \) is a Cauchy sequence satisfying \( 2 < a_n < 3 \) for all \( n \). Show that \( (\frac{a_n^3}{a_n^2 - 1}) \) is a Cauchy sequence. Your proof needs to depend directly on the definition of a Cauchy sequence and should not rely on theorem about Cauchy sequences (unless you prove the theorems as part of your solution).


15. ⋆ Prove Cauchy’s Criterion: A sequence converges if and only if it is a Cauchy sequence.

16. Prove the Absolute Convergence Test