Exercise 6.2.12 (Cantor Function) Review the construction of the Cantor set \( C \subset [0,1] \) from Section 3.1. This exercise makes use of results and notation from this discussion.

(a) Define \( f_0(x) = x \) for all \( x \in [0,1] \). Now, let

\[
 f_1(x) = \begin{cases} 
 (3/2)x & \text{for } 0 \leq x \leq 1/3 \\
 1/2 & \text{for } 1/3 < x < 2/3 \\
 (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1.
\end{cases}
\]

Sketch \( f_0 \) and \( f_1 \) over \([0,1]\) and observe that \( f_1 \) is continuous, increasing, and constant on the middle third \((1/3,2/3) = [0,1] \setminus C_1\).

(b) Construct \( f_2 \) by imitating this process of flattening out the middle third of each nonconstant segment of \( f_1 \). Specifically, set

\[
 f_2(x) = \begin{cases} 
 (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\
 f_1(x) & \text{for } 1/3 < x < 2/3 \\
 (1/2)f_1(3x - 2) + 1/2 & \text{for } 2/3 \leq x \leq 1.
\end{cases}
\]

If we continue this process, show that the resulting sequence \((f_n)\) converges uniformly on \([0,1]\).

(c) Let \( f = \lim f_n \). Prove that \( f \) is a continuous, increasing function on \([0,1]\) with \( f(0) = 0 \) and \( f(1) = 1 \) that satisfies \( f'(x) = 0 \) for all \( x \) in the open set \([0,1] \setminus C\). Recall that the “length” of the Cantor set \( C \) is 0. Somehow \( f \) manages to increase from 0 to 1 while remaining constant on a set of “length 1.”

Solution.

(a)

We observe that \( f_n \) is constant on the open set \([0,1] \setminus C_n\) for all \( n \in \mathbb{N} \).

(b) We will prove, by induction, that

\[
 |f_n(x) - f_{n-1}(x)| \leq \frac{1}{2^n}
\]

for \( x \in [0,1] \) and \( n \in \mathbb{N} \). Notice that

\[
 |f_1(x) - f_0(x)| \leq \frac{1}{2} \quad \text{and} \quad |f_2(x) - f_1(x)| \leq \frac{1}{2^2}
\]

for all \( x \in [0,1] \). Assume, for some \( k \geq 2 \), that

\[
 |f_k(x) - f_{k-1}(x)| \leq \frac{1}{2^k}, \quad x \in [0,1].
\]
Now consider the expression \(|f_{k+1}(x) - f_k(x)|\). We will apply the three-part piecewise recursive definition of \(f_n\) to prove the induction step.

Case 1: If \(0 \leq x \leq 1/3\), then
\[
|f_{k+1}(x) - f_k(x)| = |(1/2)f_k(3x) - (1/2)f_{k-1}(3x)|
\]
\[
= \frac{1}{2} \cdot |f_k(3x) - f_{k-1}(3x)|
\]
\[
\leq \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}.
\]

Case 2: If \(1/3 < x < 2/3\), then
\[
|f_{k+1}(x) - f_k(x)| = |f_k(x) - f_{k-1}(x)| = 0 < \frac{1}{2^{k+1}}.
\]

Case 3: If \(2/3 \leq x \leq 1\), then
\[
|f_{k+1}(x) - f_k(x)| = |(1/2)f_k(3x-2) + 1/2 - [(1/2)f_{k-1}(3x-2) + 1/2]|
\]
\[
= \frac{1}{2} \cdot |f_k(3x-2) - f_{k-1}(3x-2)|
\]
\[
\leq \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}
\]

This proves, by induction, that
\[
|f_n(x) - f_{n-1}(x)| \leq \frac{1}{2^n}, \quad x \in [0, 1],
\]
for all \(n \in \mathbb{N}\).

Let \(1 \leq m < n\). Then
\[
|f_m(x) - f_n(x)|
\]
\[
= |(f_m(x) - f_{m+1}(x)) + (f_{m+1}(x) - f_{m+2}(x)) + \cdots + (f_{n-1}(x) - f_n(x))|
\]
\[
\leq |f_m(x) - f_{m+1}(x)| + |f_{m+1}(x) - f_{m+2}(x)| + \cdots + |f_{n-1}(x) - f_n(x)|
\]
\[
\leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^n}
\]
\[
< \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \frac{1}{2^{m+3}} + \cdots \quad \text{(geometric series)}
\]
\[
= \frac{1}{2^m}.
\]

Now, let \(\epsilon > 0\) be given. Choose \(N \in \mathbb{N}\) such that \(\frac{1}{2^N} < \epsilon\). If \(n > m \geq N\), then by the previous calculation we have
\[
|f_m(x) - f_n(x)| < \frac{1}{2^m} \leq \frac{1}{2^N} < \epsilon.
\]

By Cauchy’s criterion for uniform convergence, the sequence \((f_n)\) of continuous functions converges uniformly to a limit function \(f\).

(c) Because each \(f_n\) is continuous and \((f_n) \to f\) uniformly, it follows that \(f\) is continuous. Since \(f_n(0) = 0\) and \(f_n(1) = 1\) for all \(n\), we have \(f(0) = 0\) and \(f(1) = 1\). Let \(0 \leq x < y \leq 1\). Since each \(f_n\) is increasing, \(f_n(x) \leq f_n(y)\). By the order limit
theorem \( f(x) \leq f(y) \), and so \( f \) is increasing. Since \( f \) is constant on each of open intervals whose union form \([0, 1] \setminus C\), it follows at \( f'(x) = 0 \) for \( x \in [0, 1] \setminus C \). \( \Box \)