Exercise 6.2.12 (Cantor Function) Review the construction of the Cantor set $C \subset [0,1]$ from Section 3.1. This exercise makes use of results and notation from this discussion.

(a) Define $f_0(x) = x$ for all $x \in [0,1]$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \leq x \leq 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

Sketch f_0 and f_1 over $[0,1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0,1] \setminus C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, set

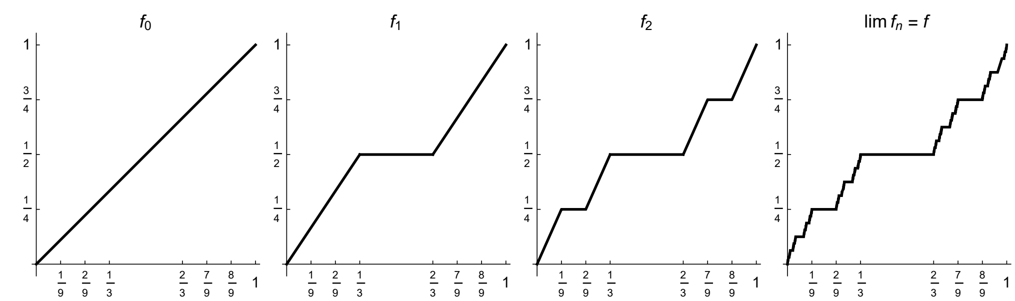
$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1. \end{cases}$$

If we continue this process, show that the resulting sequence (f_n) converges uniformly on $[0,1]$.

(c) Let $f = \lim f_n$. Prove that f is a continuous, increasing function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0,1] \setminus C$. Recall that the "length" of the Cantor set C is 0. Somehow f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

Solution.

(a)



We observe that f_n is constant on the open set $[0,1] \setminus C_n$ for all $n \in \mathbb{N}$.

(b) We will prove, by induction, that

$$|f_n(x) - f_{n-1}(x)| \leq \frac{1}{2^n}$$

for $x \in [0,1]$ and $n \in \mathbb{N}$. Notice that

$$|f_1(x) - f_0(x)| \leq \frac{1}{2} \quad \text{and} \quad |f_2(x) - f_1(x)| \leq \frac{1}{2^2}$$

for all $x \in [0,1]$. Assume, for some $k \geq 2$, that

$$|f_k(x) - f_{k-1}(x)| \leq \frac{1}{2^k}, \quad x \in [0,1].$$

Now consider the expression $|f_{k+1}(x) - f_k(x)|$. We will apply the three-part piecewise recursive definition of f_n to prove the induction step.

<u>Case 1</u>: If $0 \le x \le 1/3$, then

$$\begin{aligned}|f_{k+1}(x) - f_k(x)| &= |(1/2)f_k(3x) - (1/2)f_{k-1}(3x)| \\ &= \frac{1}{2} \cdot |f_k(3x) - f_{k-1}(3x)| \\ &\le \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}.\end{aligned}$$

<u>Case 2</u>: If $1/3 < x < 2/3$, then

$$|f_{k+1}(x) - f_k(x)| = |f_k(x) - f_{k-1}(x)| = 0 < \frac{1}{2^{k+1}}.$$

<u>Case 3</u>: If $2/3 \le x \le 1$, then

$$\begin{aligned}|f_{k+1}(x) - f_k(x)| &= \big|[(1/2)f_k(3x-2) + 1/2] - [(1/2)f_{k-1}(3x-2) + 1/2]\big| \\ &= \frac{1}{2} \cdot |f_k(3x-2) - f_{k-1}(3x-2)| \\ &\le \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}\end{aligned}$$

This proves, by induction, that

$$|f_n(x) - f_{n-1}(x)| \le \frac{1}{2^n}, \quad x \in [0,1],$$

for all $n \in \mathbb{N}$.

Let $1 \le m < n$. Then

$$\begin{aligned}&|f_m(x) - f_n(x)| \\ &= |(f_m(x) - f_{m+1}(x)) + (f_{m+1}(x) - f_{m+2}(x)) + \cdots + (f_{n-1}(x) - f_n(x))| \\ &\le |f_m(x) - f_{m+1}(x)| + |f_{m+1}(x) - f_{m+2}(x)| + \cdots + |f_{n-1}(x) - f_n(x)| \\ &\le \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^n} \\ &< \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \frac{1}{2^{m+3}} + \cdots \quad \text{(geometric series)} \\ &= \frac{1}{2^m}.\end{aligned}$$

Now, let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. If $n > m \ge N$, then by the previous calculation we have

$$|f_m(x) - f_n(x)| < \frac{1}{2^m} \le \frac{1}{2^N} < \epsilon.$$

By Cauchy's criterion for uniform convergence, the sequence (f_n) of continuous functions converges uniformly to a limit function f.

(c) Because each f_n is continuous and $(f_n) \to f$ uniformly, if follows that f is continuous. Since $f_n(0) = 0$ and $f_n(1) = 1$ for all n, we have $f(0) = 0$ and $f(1) = 1$. Let $0 \le x < y \le 1$. Since each f_n is increasing, $f_n(x) \le f_n(y)$. By the order limit theorem $f(x) \leq f(y)$, and so f is increasing. Since f is constant on each of open intervals whose union form $[0,1] \setminus C$, it follows at $f'(x) = 0$ for $x \in [0,1] \setminus C$. $\square$